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THE BRAUER GROUP OF GRADED AZUMAYA ALGEBRAS. II: GRADED GALOIS EXTENSIONS(1)

BY

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ABSTRACT. This paper continues the study of the Brauer group $B_\phi(R, G)$ of G-graded Azumaya R-algebras begun in [5]. A group $\mathrm{Galz}_\phi(R, G)$ of graded Galois extensions is constructed which always contains, and often equals, the cokernel of $B_\phi(R, G)$ modulo the usual Brauer group of R. Sufficient conditions for equality are found. The structure of $\mathrm{Galz}_\phi(R, G)$ is studied, and $\mathrm{Galz}_\phi(R, (Z/p^eZ)^r)$ is computed. These results are applied to give computations of a Brauer group of dimodule algebras constructed by F. W. Long.

In the first paper of this name ([5] = [CGO]) we introduced and studied a Brauer group of graded Azumaya algebras $B_\phi(R, G)$ for G a finite abelian group, R a commutative ring with units group $U(R)$, and ϕ: $G \times G \longrightarrow U(R)$ a fixed bimultiplicative map. When $G = Z_2$ and ϕ is nontrivial, $B_\phi(R, G)$ is the Brauer-Wall group ([19], [3], [17]), introduced to provide an appropriate image for the Clifford algebra map on quadratic spaces. When ϕ is trivial, $B(R, G)$ was studied in [14].

This paper is a sequel to [CGO] and is motivated by two problems arising in [CGO]. One problem was that of finding an appropriate group of graded Galois extensions into which $B_\phi(R, G)$ is mapped (via a map called π in [CGO]) with kernel $B(R)$, the usual Brauer group. In [CGO] this was solved by an ad hoc method of inducing a product on the image of π from that on $B_\phi(R, G)$, and computing this product explicitly as needed. This sufficed to describe $B_\phi(R, G)$ for G cyclic. In this paper we remedy the ad hoc treatment of the image of π by introducing (§1) a group $\mathrm{Galz}_\phi(R, G)$ of graded Galois extensions, with the group structure intrinsically defined, into which π maps as a homomorphism. It turns out that the group $\mathrm{Galz}_\phi(R, G)$ can be described rather explicitly in many cases, and this leads to progress on the second problem left over from [CGO], namely, describing $B_\phi(R, G)$ when G is noncyclic.

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Suppose, as we shall from §3 on, that R is a connected commutative ring, G is an abelian group of order n and exponent m, and R contains $1/n$ and a primitive mth root of unity. We show in §2 that to compute $\mathrm{Galz}_\phi(R, G)$ it suffices to do it for G a p-group. In §3 we compute $\mathrm{Galz}_\phi(R, G)$ when $G = Z_{p^e} \times \cdots \times Z_{p^e}$ (r copies) in terms of $\mathrm{Comm}(R, G)$, the group of commutative Galois extensions with group G and in terms of two $r \times r$ matrix groups with entries in Z/p^eZ: a group of skew matrices, and an orthogonal group $O^r(M)$. If p is odd and ϕ_s: $G \times G \longrightarrow U(R)$, defined by $\phi_s(\sigma, \tau) = \phi(\sigma, \tau), \phi(\tau, \sigma)$, is nondegenerate, our description becomes the exact sequence

$$1 \longrightarrow \mathrm{Comm}(R, G) \longrightarrow \mathrm{Galz}_\phi(R, G) \longrightarrow O^r_{Z/p^eZ}(M) \longrightarrow 1$$

where M is a nonsingular symmetric matrix representing ϕ_s.

In §4 we show that, with R, G as above, if $\mathrm{Pic}(R)$ is n-torsion free, then the map π maps onto $\mathrm{Galz}_\phi(R, G)$, thereby providing a description of $B_\phi(R, G)$ for a large class of noncyclic groups.

In [12], [13], F. W. Long, motivated by the theory of equivariant Brauer groups of Fröhlich and Wall [9], constructed a Brauer group $BD(R, H)$ of dimodule algebras for any finite commutative, cocommutative Hopf algebra H. We observe (§5) that when $H = RG$ and R, G are as described above, $BD(R, H) \cong B_\phi(R, G \times G)$ for an appropriate ϕ. Thus the results in this paper provide a description of Long's group, whenever R, G and $\mathrm{Pic}(R)$ are as described above and $G = \Pi_p G_p$ with $G_p = Z_{p^e} \times \cdots \times Z_{p^e}$ (n times) for any n and e, in terms of $B(R)$, $\mathrm{Comm}(R, G)$ and $O^{2n}_{Z/p^eZ}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

Assume throughout the paper that R is a commutative ring, G is a finite abelian group, and ϕ: $G \times G \longrightarrow U(R)$ is a bilinear map. Recall that if $A = \Sigma_{\sigma \in G} A_\sigma$, $hA = \bigcup_{\sigma \in G} A_\sigma$. A formula which only makes sense for elements in hA is assumed to be valid as shown only where it makes sense and is extended by linearity on all of A. The ϕ in $B_\phi(R, G)$, etc., will often be omitted. Boldface $\otimes$ denotes the graded tensor product of [CGO]. The product on G will be written multiplicatively, with identity 1.

All unexplained notation is from [CGO].

1. **The group of graded Galois extensions.** Here is the group of graded Galois extensions which is a candidate for an intrinsic description of the image of the map π of [CGO], and is, in any case, a target for π:

(1.1) DEFINITION. Let $\mathrm{Galz}_\phi(R, G)$ be the set of isomorphism classes (where isomorphisms preserve all structure) of R-algebras S with the following structures:

(i) S is a G-graded R-algebra.

(ii) S has a right G-action with G acting as grade-preserving R-algebra automorphisms.

(iii) S has a left G-action with G acting as grade-preserving R-algebra automorphisms.

These structures satisfy the following axioms, which are not independent:

(1) S is a graded Azumaya S_1-algebra (S_1 = trivially graded part of S, as in [CGO]).

(2*r*) S is a Galois extension of R on the right with group G.

(2*l*) S is a Galois extension of R on the left with group G.

(3*r*) For all a, b in hS, $ba^b = \phi(b, a)ab$, where if b is in S_σ, $a^b = a^\sigma$ denotes the right action of σ on a.

(3*l*) For all a, b in hS, $ab = \phi(a, b)b({}^b a)$, where ${}^b a$ denotes left action.

(4) For all a in S, σ in G, $a^\sigma = \phi(\sigma, a)\phi(a, \sigma)^\sigma a$.

It is easy to see that (3*r*) + (4) implies (3*l*), and that (3*r*) implies the right center of S is S_1, (3*l*) implies the left center of S is S_1, and (2) implies separability of S over S_1, so (3*r*) + (3*l*) + (2) implies (1).

Recall that a graded Azumaya R-algebra A is called fully graded [CGO, p. 309] if, for all σ, τ in G, $A_\sigma A_\tau = A_{\sigma\tau}$. Each class in the Brauer group $B(R, G)$ has a fully graded representative. For a fully graded Azumaya R-algebra A, $\pi(A) = A^{A_1} = \{x \text{ in } A \mid xa = ax \text{ for all } a \text{ in } A_1\}$.

In the rest of this section we prove:

(1.2) THEOREM. *The set* Galz(R, G) *is a group, the map* π *is a homomorphism into* Galz(R, G), *and hence* (*by* [CGO, (3.13)]) *there is an exact sequence*

$$0 \longrightarrow B(R) \longrightarrow B(R, G) \longrightarrow \mathrm{Galz}(R, G).$$

PROOF. We begin by showing that for a fully graded Azumaya R-algebra A, $\pi(A)$ is an element of Galz(R, G).

The grading on $\pi(A)$ is that induced from the inclusion $\pi(A) \subseteq A$.

The right action of G on $\pi(A)$ was described in [CGO, (p. 311)]: if b_σ^i in $A_{\sigma^{-1}}$, c_σ^i in A_σ are elements satisfying $\Sigma_i b_\sigma^i c_\sigma^i = 1$, then defining for a in A^{A_1}, $a^\sigma = \phi(\sigma, a)\, \Sigma_i b_\sigma^i a c_\sigma^i$, we showed [CGO, (3.4)] that this action makes $\pi(A)$ into a Galois extension of R with group G. From the definition it is clear that the right action of G preserves the grading by G on $\pi(A)$. We also showed [CGO, (3.3)] that

$$\pi(A) \subseteq \{a \text{ in } A \mid xa^x = \phi(x, a)ax \text{ for all } x \text{ in } hA\};$$

the converse is obvious. Thus $\pi(A)$ satisfies (3*r*) and (2*r*) for the right G-structure.

The left G-structure arises from the fact that if A is a graded Azumaya R-algebra, so is $A^{\#}$, and $\pi(A^{\#}) = (\pi(A))^{\#}$ is also a graded R-algebra with a right G-structure. The grading on $A^{\#}$ is the same as that on A.

Define the left G-structure on πA by $({}^{\sigma}a)^{\#} = (a^{\#})^{\sigma^{-1}}$. Then, since $\pi(A)^{\#}$ satisfies (2r), $\pi(A)$ satisfies (2l). Since $\pi(A)^{\#} = \{a^{\#} | x^{\#}a^{\#x} = \phi(x, a)a^{\#}x^{\#}$ for all x in $hA\}$, therefore (recalling that $x^{\#}a^{\#} = \phi(x, a)(ax)^{\#}$)

$$\pi(A) = \{a | ax = \phi(a, x)x({}^{x}a) \text{ for all } x \text{ in } hA\},$$

hence satisfies (3l).

If $d_{\sigma}^{i} \in A_{\sigma}$, $e_{\sigma}^{i} \in A_{\sigma^{-1}}$, $\Sigma d_{\sigma}^{i\#}e_{\sigma}^{i} = \Sigma e_{\sigma}^{i}d_{\sigma}^{i}\phi(\sigma, \sigma^{-1}) = 1$, then

$$({}^{\sigma}a)^{\#} = (a^{\#})^{\sigma^{-1}} = \phi(\sigma^{-1}, a)\sum d_{\sigma}^{i\#}a^{\#}e_{\sigma}^{i\#}$$

$$= \phi(\sigma^{-1}, a)\sum \phi(d_{\sigma}^{i}, a)\phi(d_{\sigma}^{i}a, e_{\sigma}^{i})(e_{\sigma}^{i}ad_{\sigma}^{i})^{\#}$$

so

$${}^{\sigma}a = \phi(\sigma^{-1}, a)\phi(\sigma, a)(\sigma a, \sigma^{-1})\sum e_{\sigma}^{i}ad_{\sigma}^{i},$$

but $a^{\sigma} = \phi(\sigma, a)\Sigma e_{\sigma}^{i}a(d_{\sigma}^{i}\phi(\sigma, \sigma^{-1}))$ so ${}^{\sigma}a = \phi(\sigma^{-1}, a)\phi(a, \sigma^{-1})a^{\sigma}$, verifying (4). Thus $\pi(A)$ satisfies all the axioms.

The product on Galz(R, G) will be motivated by that induced by π, which we now describe. For A, B in $B(R, G)$,

$$\pi(A \otimes B) = \Big\{\sum a_i \otimes b_i \Big| \Big(\sum a_i \otimes b_i\Big)(x \otimes y) = \sum (x \otimes y)(a_i \otimes b_i) \text{ for all } x \otimes y \text{ in } A_{\sigma} \otimes B_{\sigma^{-1}}\Big\}.$$

Since $\pi(A \otimes B) \subseteq \pi(A) \otimes \pi(B)$,

$$\Big(\sum a_i \otimes b_i\Big)(x \otimes y) = \sum (x \otimes y)(a_i \otimes b_i) = \sum \phi(y, a_i)xa_i \otimes yb_i$$
$$= \sum \phi(b_i, x)a_ix \otimes b_iy = \sum \phi(b_i, x)\phi(x, a_i)^{-1}xa_i^{x} \otimes \phi(b_i, y)y^{y}b_i.$$

Since $\phi(\sigma, xy) = 1 = \phi(xy, \sigma)$ for all σ, it follows that $\Sigma xa_i^{x} \otimes y^{y}b_i = \Sigma xa_i \otimes yb_i$, so $(x \otimes y)(\Sigma a_i^{\sigma} \otimes {}^{\sigma^{-1}}b_i - \Sigma a_i \otimes b_i) = 0$ for all $x \otimes y$ in $A_{\sigma} \otimes A_{\sigma^{-1}}$. Choose z_j, w_j in $A_{\sigma^{-1}}$, A_{σ}, with $\Sigma z_jw_j = 1$, and x_k, y_k in A_{σ}, $A_{\sigma^{-1}}$ with $\Sigma x_ky_k = 1$; then

$$0 = \sum_{j,k} (z_j \otimes x_k)(w_j \otimes y_k)\Big(\sum a_i^{\sigma} \otimes {}^{\sigma^{-1}}b_i - \sum a_i \otimes b_i\Big)$$
$$= \sum a_i^{\sigma} \otimes {}^{\sigma^{-1}}b_i - \sum a_i \otimes b_i.$$

Thus $\pi(A \otimes B) = \{\Sigma a_i \otimes b_i | \Sigma a_i^\sigma \otimes {}^{\sigma^{-1}}b_i = \Sigma a_i \otimes b_i$ for all σ in $G\}$. So we define $S \cdot T = \{\Sigma s_i \otimes t_i$ in $S \otimes T | \Sigma s_i^\sigma \otimes {}^{\sigma^{-1}}t_i = \Sigma s_i \otimes t_i$ for all σ in $G\}$.

The identity of $\mathrm{Galz}(R, G)$ is the image of the identity of $B(R, G)$, namely GR = the set of all functions from G to R with pointwise addition and multiplication and with trivial grading [CGO, (3.7)]. The inverse of S in $\mathrm{Galz}(R, G)$ is $S^\#$, with grading induced from S and left and right actions by G given by $(s^\#)^\sigma = ({}^{\sigma^{-1}}s)^\#$, ${}^\sigma(s^\#) = (s^{\sigma^{-1}})^\#$ [CGO, p. 314].

We must show that $\mathrm{Galz}(R, G)$ is closed under multiplication and that the axioms for a group hold. Once we do that it will be clear that π is a homomorphism and, given [CGO, §3], the theorem will be proved.

Closure. The right action on $\pi(A \otimes B)$ is induced from the right action on B: let $c_\sigma^i \in B_{\sigma^{-1}}$, $d_\sigma^i \in B_\sigma$ with $\Sigma c_\sigma^i d_\sigma^i = 1$. Then

$$\begin{aligned}(a \otimes b)^\sigma &= \phi(\sigma, ab)\textstyle\sum(1 \otimes c_\sigma^i)(a \otimes b)(1 \otimes d_\sigma^i)\\ &= \phi(\sigma, ab)\phi(\sigma^{-1}, a)\textstyle\sum a \otimes c_\sigma^i b d_\sigma^i\\ &= \textstyle\sum a \otimes \phi(\sigma, b) c_\sigma^i b d_\sigma^i = a \otimes b^\sigma.\end{aligned}$$

So for S, T in $\mathrm{Galz}_\phi(R, G)$ define the right action on $S \cdot T$ to be that on T. Similarly, the left action on $\pi(A \otimes B)$ is induced from the left action on A, so define the left action of G on $S \cdot T$ to be that on S.

Give $S \cdot T$ the grading induced from the product grading on $S \otimes T$. To check that $S \cdot T$ is in $\mathrm{Galz}(R, G)$ it suffices to show that

2. $S \cdot T$ is a Galois extension of R on the right with group G and on the left with group G, both of which preserve grading;
3. for all a, b in $h(S \cdot T)$, $ba^b = \phi(b, a)ab$;
4. for all a in $h(S \cdot T)$, σ in G, $a^\sigma = \phi(\sigma, a)\phi(a, \sigma)^\sigma a$.

Proof of 2. If S, T are in $\mathrm{Galz}(R, G)$ then $S \otimes T$ is a Galois extension of R with group $G \times G$, $(\sigma, \tau)(s \otimes t) = s^\sigma \otimes {}^\tau t$. For if $\{s_i, r_i\}$ are Galois elements for the right action of G on S (i.e., $\Sigma_i s_i r_i^\sigma = \delta_{1,\sigma}$ for all $\sigma \in G$) and $\{t_j, u_j\}$ are Galois elements for the left action of G on T, then it is quickly checked that

$$\{s_i \otimes t_j, \phi^{-1}(t_j, r_i) r_i \otimes u_j\}$$

is a set of Galois elements for $S \otimes T$; hence condition 1.3b of [4] holds for the action of $G \times G$ on $S \otimes T$. It follows easily by [4, 2.2] that $S \otimes T$ is a Galois extension of $S \cdot T$ with group $D = \{(\sigma, \sigma^{-1}) \in G \times G\}$, and, hence, by [4, 1.6], that there exists an element $c = \Sigma_k c_k \otimes d_k$ in $S \otimes T$ with $\Sigma_{\sigma \in G} \Sigma_k c_k^\sigma \otimes {}^{\sigma^{-1}}d_k = 1 \otimes 1$, i.e., whose trace is one. By taking homogeneous components we can choose c to be in $S_1 \otimes T_1$.

Then, following [4, p. 23], let $\{v_j, w_j\}$ be Galois elements for the right action of G on T, and $\{s_i, r_i\}$ be as above. Consider $\{x_{ij}, y_{ij}\}$ in $S \cdot T$ as follows:

$$x_{ij} = \sum_{\sigma} \left((s_i^{\sigma} \otimes {}^{\sigma^{-1}} v_j) \left(\sum_k c_k^{\sigma} \otimes {}^{\sigma^{-1}} d_k \right) \right),$$

$$y_{ij} = \sum_{\sigma} \phi^{-1}(v_j, r_i)(r_i^{\sigma} \otimes {}^{\sigma^{-1}} w_j).$$

Each of these is the trace over D of an element in $S \otimes T$ so is in $S \cdot T$. It is easily verified that $\{x_{ij}, y_{ij}\}$ are Galois elements for the right action of G on $S \otimes T$; hence (2r) holds.

A similar argument holds to show (2l).

3. Let $a \otimes b$, $c \otimes d$ be homogeneous elements of $S \cdot T \subseteq S \otimes T$. Then

$$\phi(ab, cd)(c \otimes d)(a \otimes b) = \phi(ab, cd)\phi(d, a)ca \otimes db$$

$$= \phi(ab, cd)\phi(d, a)ac^a\phi(a, c)^{-1} \otimes bd^b\phi(b, d)^{-1}$$

$$= \phi(b, c)\phi(a, d)\phi(d, a)ac^a \otimes bd^b$$

$$= \phi(a, d)\phi(d, a)(a \otimes b)(c^a \otimes d^b) = \phi(a, d)\phi(d, a)(a \otimes b)(c \otimes {}^a d^b)$$

$$= (a \otimes b)(c \otimes d^{ab}) = (a \otimes b)(c \otimes d)^{(a \otimes b)}$$

and similarly for homogeneous elements of $S \cdot T$ which are sums of tensors in $S \otimes T$.

4. $$(a \otimes b)^{\sigma} = a \otimes b^{\sigma} = \phi(\sigma, b)\phi(b, \sigma)a \otimes {}^{\sigma}b$$

$$= \phi(\sigma, b)\phi(b, \sigma)\phi(a, \sigma)\phi(\sigma, a)^{\sigma}a \otimes b$$

$$= \phi(\sigma, ab)\phi(ab, \sigma)^{\sigma}(a \otimes b)$$

and similarly for sums of tensors.

So $\mathrm{Galz}(R, G)$ is closed under multiplication.

Associativity. We have $S(TU) = (S \otimes (T \otimes U)^D)^D$. Since $T \otimes U$ is a Galois extension of $(T \otimes U)^D$ with group D, every element of $(T \otimes U)^D$ is the trace of some element of $T \otimes U$, etc. Thus a typical element of $S(TU)$ is of the form

$$\sum_i \sum_{\sigma} s_i^{\sigma} \otimes {}^{\sigma^{-1}} \left(\sum_j \sum_{\tau} t_{ij}^{\tau} \otimes {}^{\tau^{-1}} u_{ij} \right) = \sum_{ij} \sum_{\tau} \left(\sum_{\sigma} s_i^{\sigma} \otimes {}^{\sigma^{-1}} t_{ij} \right)^{\tau} \otimes {}^{\tau^{-1}} u_{ij},$$

a sum of elements in $((S \otimes T)^D \otimes U)^D = (ST)U$.

Identity. The identity of $\mathrm{Galz}(R, G)$ is GR = functions from G to R with trivial grading. On GR the left and right G-structures coincide, and if S is any graded Galois extension $S \otimes GR = S \otimes GR$ since GR is trivially graded. Thus as a graded algebra and right Galois extension $S \cdot GR = S$ by the standard argument [CGO, p. 313]; since the left G-structure on $S \cdot GR$ is the same as that on S, GR is the identity.

Inverse. The inverse of S is $S^{\#}$, as described above. Consider the map $f\colon (S \otimes S^{\#})^D \longrightarrow GR$, defined by linearity and $f(xy^{\#})(\tau) = x({}^{\tau}y)$. The image of $f(xy^{\#})$ is in fact in R since

$$(x^{\tau}y)^{\sigma} = x^{\sigma} \cdot {}^{\tau}y^{\sigma} = f(x^{\sigma} \otimes (y^{\sigma})^{\#})(\tau)$$

$$= f(x^{\sigma} \otimes {}^{\sigma^{-1}}(y^{\#}))(\tau) = f(x \otimes y^{\#})(\tau) = x^{\tau}y;$$

f is an algebra map since

$$f(x \otimes y^{\#})(z \otimes w^{\#})(\tau) = f(xz \otimes y^{\#}w^{\#}\phi(y, z))(\tau)$$

$$= f(xz \otimes \phi(y, w)(wy)^{\#}\phi(y, z))(\tau)$$

$$= f(xz \otimes (wy)^{\#})(\tau) \quad \text{since } zw \text{ is in } R \quad (f(z \otimes w^{\#})(1) = zw)$$

$$= xz\,{}^{\tau}(wy) = xz\,{}^{\tau}w\,{}^{\tau}y$$

$$= (x\,{}^{\tau}y)(z\,{}^{\tau}w) \quad \text{since } z\,{}^{\tau}w \text{ is in } R$$

$$= f(x \otimes y^{\#})(\tau) \cdot f(z \otimes w^{\#})(\tau).$$

By [10, Corollary 2, p. 5], it follows that $(S \otimes S^{\#})^D = GR$, and $S^{\#}$ is the inverse of S in $\mathrm{Galz}(R, G)$. This completes the proof.

2. **Reduction to p-groups.** The next two sections are devoted to a study of $\mathrm{Galz}_{\phi}(R, G)$ for a given fixed bilinear map $\phi\colon G \times G \longrightarrow U(R)$. In this section we prove

(2.1) THEOREM. *Let $G = H \times J$ where the orders of H and J are relatively prime. Let ϕ_H, ϕ_J be the restrictions of ϕ to H, J. Then* $\mathrm{Galz}_{\phi}(R, G) = \mathrm{Galz}_{\phi_H}(R, H) \times \mathrm{Galz}_{\phi_J}(R, J)$.

(2.2) COROLLARY. *Let $G = \Pi G_p$ be the decomposition of G into its p-primary components, and let ϕ_p be ϕ on $G_p \times G_p$. Then*

$$\mathrm{Galz}_{\phi}(R, G) = \prod_p \mathrm{Galz}_{\phi_p}(R, G_p).$$

The equality of (2.1) will be obtained by decomposing any S in

$\mathrm{Galz}_\phi(R, G)$ as

(2.3) $$S \cong S^J \otimes_R S^H.$$

To prove (2.1) it suffices to show

(2.4) The decomposition (2.3) is as R-algebras and G-modules.

(2.5) The algebras S^J and S^H are in $\mathrm{Galz}_{\phi_H}(R, H)$, $\mathrm{Galz}_{\phi_J}(R, J)$, respectively.

(2.6) The decomposition (2.3) gets along with the group structures on $\mathrm{Galz}_\phi(R, G)$, etc.

PROOF OF (2.4). Since S is a Galois extension of R with group $G = H \times J$ it is standard that $S \cong S^J \otimes_R S^H$ as R-modules and G-modules. The only nontrivial aspect of (2.4) is in showing that (2.3) is a decomposition as R-algebras, that is, that elements of S^J and S^H commute with each other. To do this we look at the R-module generated by $\{xy - yx | x \in S^J, y \in S^H\} \subseteq S$. To show it is zero and (2.4) holds it suffices to show it when we replace R by a faithfully flat extension, namely, the direct sum of the stalks on the Boolean spectrum of R, a faithfully flat extension of R [18, 2.9]. Since these stalks are connected [18, 2.13] it suffices to assume R is connected. In that case, S, being a Galois extension of R with abelian group G, is a central Galois extension of T, the center of S, with group L [11, Proposition 8]. Then $S = \Sigma \bigoplus J_\sigma$ with $J_1 = T$, J_σ rank one projective T modules, for σ in L [11]. Here $J_\sigma = \{a \in S | ax^\sigma = xa\}$. The noncommutativity of S is expressed by a skew nondegenerate bilinear form ψ on L: for a in J_σ, b in J_τ, $ba = \psi(\tau, \sigma)ab$ [8], [6]. Since $L \cap H$ and $L \cap J$ have relatively prime orders, $L \cap H$ and $L \cap J$ are orthogonal with respect to ψ, so ψ must be nondegenerate on both $L \cap H$ and $L \cap J$. We have $J_\sigma J_\tau = J_{\sigma\tau}$ [11], so we may write $S = (\Sigma_{\sigma \in (H \cap L)} J_\sigma) \otimes_T (\Sigma_{\sigma \in (J \cap L)} J_\sigma) = S_H \otimes_T S_J$. Clearly $S_J \subseteq S^{S_H}$, the commutator of S_H in S; in fact, $S_J = S^{S_H}$: for S is an Azumaya T-algebra and S_H and S_J both have center T by the nondegeneracy of ψ on $L \cap H$ and $J \cap H$, and are contained in each other's commutators; [2, (3.3)] applies to yield $S_J = S^{S_H}$, $S_H = S^{S_J}$. Now for x in S, σ in $H \cap L$, a in $J_\sigma \subseteq S_H$, we have $ax^\sigma = xa$. If x is in S^H, $x^\sigma = x$ so $ax = xa$. So $S^H \subseteq S^{S_H} = S_J$. Similarly, $S^J \subseteq S_H$. So elements of S^H and of S^J commute with each other.

PROOF OF (2.5). The difficulty in showing that if S is in $\mathrm{Galz}_\phi(R, G)$ then S^J is in $\mathrm{Galz}_{\phi_H}(R, H)$ lies in showing that the gradings of S^J are in H. So we begin (2.5) with some lemmas on the gradings of S in $\mathrm{Galz}_\phi(R, G)$.

(2.7) LEMMA. *Let R be connected, S in* $\mathrm{Galz}_\phi(R, G)$. *For each σ in $K = \{\sigma \in G | S_\sigma \neq 0\}$:*

(i) $S_\sigma = \{a \in S | ax^\sigma = \phi(a, x)xa\}$ *is a rank one projective* S_1*-module,*
(ii) $S_\sigma S_\tau = S_{\sigma\tau}$.

PROOF OF (2.7). Since S is in $\mathrm{Galz}_\phi(R, G)$, S is a graded Azumaya S_1-algebra, by (1.1).

Let ${}^eS = S^\# \otimes_{S_1} S$ [CGO, (1.3)] and view S as an S-S bimodule, $S(1, \sigma)$, by the usual left S-module action of S on S and a right action by $s \cdot x = sx^\sigma$. Then the Morita equivalence [CGO, 2.8] between S_1-modules and right ${}^eS$-modules shows that $J_\sigma = \{a \in S | ax^\sigma = \phi(ax)xa\} = {}^SS(1, \sigma)$ is a rank one projective S_1-module. Now $J_\sigma = S_\sigma$. For clearly $S_\sigma \subseteq J_\sigma$; on the other hand, if a is in S_ρ, $ax^\rho = \phi(a, x)xa$ for all $x \in S$, so $a \in J_\sigma \cap S_\rho$ iff $x^\rho = x^\sigma$ for all $x \in S$. But S is a Galois extension of R with group G, so if $x^\rho = x^\sigma$ for all x in S, $\rho = \sigma$. Thus $J_\sigma = S_\sigma$. That $S_\sigma S_\tau = J_\sigma J_\tau = J_{\sigma\tau} = S_{\sigma\tau}$ follows by tracing through the maps in the isomorphism of $J_\sigma \otimes J_\tau$ with $J_{\sigma\tau}$ in [15, Lemma 5] (as was observed in [7, Lemma 1]).

(2.8) LEMMA. *Assume* R *is connected. If* $K = \{\sigma \in G | S_\sigma \neq 0\}$ *then* K *is a group and* S *is a Galois extension of* S_1 *with group* K.

PROOF OF (2.8). From (2.7) (ii) it is clear that K is a submonoid of G, so since G is finite, K is a group. Now note that $S_1 \subseteq S^K$. For if σ is in K, a is in S_σ, x is in S_1, then $ax^a = \phi(a, x)xa$. But S_1 is in the center of S (use the same formula with x arbitrary and a in S_1), so $ax = xa$, so $a(x^a - x) = 0$. Choose a_i in S_σ, b_i in $S_{\sigma-1}$ with $\Sigma b_i a_i = 1$; then $0 = \Sigma b_i a_i(x^a - x) = x^a - x$. Thus $S_1 \subseteq S^K$. Now, since S is a Galois extension of S^K with group K, S^K is a direct summand of S, S is a projective S^K-module [4] and $\mathrm{rank}_{S_1} S = [K : 1] = \mathrm{rank}_{S^K}(S)$. Since S^K is S_1-projective, S_1 is an S_1-direct summand of S^K. Thus $S^K = S_1$.

PROOF OF (2.5). We check the conditions of (1.1) for $T = S^J$. The right action of H on T is clear. We must show that the set of gradings of T is contained in H.

We first observe that T is a projective T_1-module. For this it suffices to assume R is local. Now T is a Galois extension of R with group $H = G$ restricted to T, and T_1 is the fixed ring of K restricted to T since $T_1 = T \cap S_1$, by (2.8). So T is a Galois extension of T_1 with group $K \cap H$; hence T is T_1-projective, and $\mathrm{rank}_{T_1}(T)$ divides the order of H at each prime ideal of T_1.

To show that the set of gradings of T is contained in H it suffices to assume T_1 is local. We have $T = \Sigma_{\sigma \mathrm{in} L} \bigoplus T_\sigma$, where $L = \{\sigma \in G | T_\sigma \neq 0\}$, so the T_σ, being T_1-submodules of T, are T_1-projective. The multiplication map $T \otimes_{T_1} S_1 \longrightarrow TS_1$ is an isomorphism since both are Galois extensions of

S_1 with group $H \cap K \cong K/(J \cap K)$ [4, (3.4)]. This map sends $T_\sigma \otimes_{T_1} S_1$ one-one into S_σ. Since S_σ is a rank one projective S_1-module, T_σ must be a rank one projective T_1-module. We show $T_\sigma \cdot T_\tau \neq 0$, which will suffice to show that L is a group whose order $= \text{rank}_{T_1}(T)$ divides the order of H; hence $L \subseteq H$.

To show $T_\sigma \cdot T_\tau \neq 0$ it suffices to show that $T_\sigma S_1 \cdot T_\tau S_1 \neq 0$, since S_1 is in the center of S. But $T_\sigma S_1$ and $T_\tau S_1$ are rank one projective S_1-submodules of the rank one projectives S_σ and S_τ. Since S_1 is semilocal, we may pick free bases $S_\sigma = S_1 x$, $S_\tau = S_1 y$, $T_\sigma S_1 = S_1 cx$, $T_\tau S_1 = S_1 dy$, $x, y \in S$, c, d in S_1; then $S_\sigma S_\tau = S_{\sigma\tau} = S_1 xy$. Since x, y, cx, dy and xy are bases for free S_1-modules of rank 1, none can be divisors of zero in S_1. Thus $T_\sigma S \cdot T_\tau S_1 = S_1 cdxy \neq 0$, for as is easily seen, $cdxy$ cannot be a zero divisor. Thus $L \subseteq H$.

To show that T is a left H-module observe that since T is graded by H, for all a in T, σ in J, $a^\sigma = {}^\sigma a$, by (1.1), (4). Thus $S^J = T \subseteq {}^J S$. The left-right symmetry of the argument thus far gives the opposite inclusion. Thus $T = {}^J S$ and has a natural left H-action induced from the left action of G.

It is clear then that $(2r)$ and $(2l)$ of (1.1) hold for T, and that $(3r)$, $(3l)$ and (4) hold by inheritance from S. That proves (2.5).

PROOF OF (2.6). With $G = H \times J$ as above, let S, T be in $\text{Galz}_\phi(R, G)$. Then $S = S^J \otimes S^H$, $T = T^J \otimes T^H$ with S^J, T^J in $\text{Galz}_\phi(R, H)$, S^H, T^H in $\text{Galz}_{\phi_J}(R, J)$, by (2.4) and (2.5). We have $S \cdot T = S^J \cdot T^J \otimes S^H \cdot T^H$. For H and J are mutually orthogonal with respect to ϕ, so the usual "switch" map

$$S \otimes T = (S^J \otimes S^H) \otimes (T^J \otimes T^H) \cong (S^J \otimes T^J) \otimes (S^H \otimes T^H)$$

is an R-algebra isomorphism. Then if DG, etc., is the kernel of the multiplication map from $G \times G$ to G, acting on $S \otimes T$ by $(\sigma^{-1}, \sigma)(s \otimes t) = s^\sigma \otimes {}^\sigma t$ (cf. proof of (1.2)) it is easy to see that if $(S \otimes T)^{DG} = S \cdot T$ is the fixed ring under the action of DG, then $(S \otimes T)^{DG} \cong (S^J \otimes T^J)^{DH} \otimes (S^H \otimes T^H)^{DJ}$. This completes the proof of (2.6) and hence of (2.1).

3. **Matrix computations.** In view of Corollary (2.2) in order to describe $\text{Galz}_\phi(R, G)$, it is enough to do it for p-groups. This section will be devoted to a description of $\text{Galz}_\phi(R, G)$ for G a p-group which is a finite product of cyclic groups of equal exponent, i.e., groups which are free modules of finite rank over $Z/p^e Z$. Before restricting ourselves to such G we first obtain a general remark about elements of $\text{Galz}(R, G)$ under the following assumption, which will remain in force throughout this section:

(3.1) Assume G is a finite abelian group of order n and exponent m, and R is a connected commutative ring containing $1/n$ and a primitive mth root of unity.

Under the assumption (3.1) $G^* = \mathrm{Hom}(G, U(R)) \cong G$, and we may decompose any abelian Galois extension, just as in classical Kummer Theory of fields, into a direct sum of R-submodules, indexed by G^*, on each of which each element of G acts as multiplication by a root of unity [6].

Let S be in $\mathrm{Galz}(R, G)$. Then S is graded in three ways: by G, and by $G^* = \mathrm{Hom}(G, U(R))$ in two different ways, arising from the left and right actions of G on S, respectively, as follows: Since S is, with respect to each of the G-actions, a Galois extension of R with group G, $S = \Sigma_\chi S_\chi = \Sigma_\chi {}_\chi S$ with respect to the G^*-gradings, where

$$S_\chi = \{s \text{ in } S \mid s^\sigma = \chi(\sigma)s\}, \quad {}_\chi S = \{s \text{ in } S \mid {}^\sigma s = \chi(\sigma)s \text{ for all } \sigma \text{ in } G\}.$$

(3.2) LEMMA. *Each* S_χ *is a* ${}_\psi S$ *and each is homogeneous with respect to* G.

PROOF. Since the left action by G, the right action by G, and the grading by G all commute with each other, we may view S as an $RG \otimes RG \otimes RG^*$-module, hence as an $RG^* \otimes RG^* \otimes RG$-comodule, i.e., as graded by $G^* \times G^* \times G$. Thus

$$S = \sum_{(\psi,\chi,\sigma)} S_{(\psi,\chi,\sigma)}$$

($S_{(\psi,\chi,\sigma)}$ = elements homogeneous of degree (ψ, χ, σ) on $G^* \times G^* \times G$)

and, in particular, each $S_\chi = \Sigma_{(\psi,\sigma)} S_{(\psi,\chi,\sigma)}$; each ${}_\psi S = \Sigma_{(\chi,\sigma)} S_{(\psi,\chi,\sigma)}$. But each S_χ, respectively ${}_\psi S$, is a rank one projective R-module, so that (since R is connected) there can be only one summand. Thus $S_\chi = S_{(\psi,\chi,\sigma)}$ for some ψ, σ; also ${}_\psi S = S_{(\psi,\chi,\sigma)}$ for some χ, σ. Thus if $S_\chi = S_{(\psi,\chi,\sigma)}$, $S_\chi = {}_\psi S \subseteq S_\sigma$.

We now assume that G is a p-group. We wish to compute $\mathrm{Galz}_\phi(R, G)$. This is relatively complicated in general, but becomes manageable in case

(3.3) G is a product of cyclic groups all of the same order $m = p^e$.

For in that case we can map $\mathrm{Galz}(R, G)$ into a group of orthogonal matrices over Z/p^eZ. We now describe how this is done.

Let $G = \Pi_{i=1}^r Z_i$ with $Z_i = \langle \sigma_i \rangle$, cyclic of order p^e, and $G^* = \Pi_{i=1}^r Z_i^*$ with $Z_i^* = \langle \chi_i \rangle$, where, if ζ is a fixed mth root of unity, $\chi_i(\sigma_i) = \zeta$. Let S be in $\mathrm{Galz}_\phi(R, G)$.

From [6] and (3.1) we can write $S = \Sigma_{\chi \in G^*} S_\chi$ where G acts on s in S_χ by $s^\tau = \chi(\tau)s$. By (3.2), $S_\chi = S_{(\psi,\chi,\sigma)}$, that is, ${}^\tau s = \psi(\tau)s$ and s is homogeneous of grade σ. By [6] all of these gradings respect the multiplication in S; hence

$$S_{(\psi_1,\chi_1,\sigma_1)}\cdot S_{(\psi_2,\chi_2,\sigma_2)} = S_{(\psi_1\psi_2,\chi_1\chi_2,\sigma_1\sigma_2)}.$$

We will associate to S some $r \times r$ matrices with entries in Z/mZ.

Notation. Let $\chi = (\chi_1, \cdots, \chi_r)$, $\sigma = (\sigma_1, \cdots, \sigma_r)$ and, if β is a vector $(\beta_1, \cdots, \beta_r)$ in $(Z/mZ)^r$, write $\chi^\beta = \chi_1^{\beta_1}\chi_2^{\beta_2}\cdots\chi_r^{\beta_r}$, and similarly for σ^β.

For each i, $S_{\chi_i} = S_{(\chi^{p_i},\chi^{\epsilon_i},\sigma^{\alpha_i})}$ where p_i, ϵ_i, α_i are (row) vectors in $(Z/mZ)^r$, $\epsilon_i = (0, \cdots, 1, \cdots 0)$(1 in ith entry). Now S is generated as an algebra by the S_{χ_i}, so the left G-action on S is described by the $r \times r$ matrix

$$P = \begin{pmatrix} p_1 \\ \vdots \\ p_r \end{pmatrix}$$

of left G^*-gradings of the S_{χ_i}, $i = 1, \cdots, r$, and similarly the G-grading on S is described by the $r \times r$ matrix

$$A = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix}$$

of G-gradings of the S_{χ_i}. The corresponding matrix of right G^*-gradings is

$$\begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_r \end{pmatrix} = I.$$

Let u_i be a typical element of S_{χ_i} for $i = 1, \cdots, r$. From [6] (cf. proof of (2.4) above) we know that $u_i u_j = \delta(\chi_i, \chi_j)u_j u_i$ where δ: $G^* \times G^* \longrightarrow U(R)$ is a skew bilinear map; δ thus describes the way elements of S (fail to) commute with each other. By bilinearity, $\delta(\chi_i, \chi_j) = \zeta^{d_{ij}}$ and $D = (d_{ij})$ is clearly an $r \times r$ skew-symmetric matrix with entries in Z/mZ.

From bilinearity of ϕ we have $\phi(\sigma_i, \sigma_j) = \zeta^{\phi_{ij}}$ where $\phi = (\phi_{ij})$ is an $r \times r$ matrix in Z/mZ. Set $M = \phi + \phi'$ where $'$ means transpose.

The structure of S as an element of $\mathrm{Galz}_\phi(R, G)$ defines some relations among P, A, D, ϕ, M.

The relations:

(1) $u_i u_j^{u_i} = \phi(u_i, u_j)u_j u_i$,

(2) $u_i u_j = \delta(\chi_i, \chi_j)u_j u_i$ with δ skew bilinear,

(3) $u_i^\sigma = \phi(\sigma, u_i)\phi(u_i, \sigma)^\sigma u_i$,

(4) S is a left Galois extension with group G, for all i, j, σ

are necessary in order that S be in $\mathrm{Galz}_\phi(R, G)$.

Condition (4) is equivalent to the nonsingularity of P.

Conditions (1) and (2) combine to yield:

$$u_i u_j^{u_i} = \phi(u_i, u_j)\delta(\chi_j, \chi_i)u_i u_j, \quad \text{i.e.,}$$

$$\langle \chi^{\epsilon_j}, \sigma^{\alpha_i} \rangle = \phi(\sigma^{\alpha_i}, \sigma^{\alpha_j})\delta(\chi_j, \chi_i), \quad \text{i.e.,}$$

$$\zeta^{\alpha_i \epsilon_j'} = \zeta^{\alpha_i \phi \alpha_j'} \zeta^{d_{ji}}, \quad \text{i.e.,}$$

$$d_{ij} + \alpha_i \epsilon_j' = \alpha_i \phi \alpha_j', \quad \text{i.e.,}$$

$$D + AI = A\phi A'.$$

Condition (3) becomes

$$\langle \chi^{\epsilon_i}, \sigma_j \rangle = \phi(\sigma_j, \sigma^{\alpha_i})\phi(\sigma^{\alpha_i}, \sigma_j) \langle \chi^{\rho_i}, \sigma_j \rangle, \quad \text{i.e.,}$$

$$\epsilon_i \epsilon_j' = \epsilon_j \phi \alpha_i' + \alpha_i \phi \epsilon_j' + \rho_i \epsilon_j', \quad \text{i.e.,}$$

$$I = A\phi' + A\phi + P.$$

Summing up, conditions (1)–(4) yield:

$$D = A\phi A' - A \quad \text{is skew,}$$

$$P = I - AM \quad \text{is nonsingular.}$$

(3.4) THEOREM. *With* $\dot{R}, G$ *as in* (3.1), (3.3), $\mathrm{Galz}(R, G)$ *is described by the following two exact sequences*:

(3.4.1) $$1 \longrightarrow \mathrm{Galz}^0_\phi(R, G) \longrightarrow \mathrm{Galz}_\phi(R, G) \xrightarrow{\beta} O^r_{Z/mZ}(M),$$

(3.4.2) $$1 \longrightarrow \mathrm{Comm}(R, G) \longrightarrow \mathrm{Galz}^0_\phi(R, G) \xrightarrow{\nu} \mathrm{Skew}^r_{Z/mZ}(M),$$

where $O^r_{Z/mZ}(M) = \{P \in GL_r(Z/mZ) \mid P'MP = M\}$ *and* β, *defined by* $\beta(S) = P$, *is an antihomomorphism,* $\mathrm{Skew}^r_{Z/mZ}(M) = \{A \in M_r(Z/mZ) \mid AM = 0$ *and* $A' = -A\}$ *and* ν, *defined by* $\nu(S) = A$, *is a homomorphism.*

If $M = 0, \beta = 1$.

If $p \neq 2, \nu$ *is onto*; *if* M *is nonsingular,* $\nu = 0$ *and* β *is onto.*

If $p = 2, \mathrm{Im}\,\nu = \{A \in \mathrm{Skew}^r(M) \mid (A\phi A' - A)_{ii} = 0$ *for all* $i = 1 \cdots r\}$; *if* M *is nonsingular,* $\nu = 0$ *and* $\mathrm{Im}\,\beta = \{P \in O^r(M) \mid [(I-P)M^{-1}\phi M^{-1}(I-P) - (I-P')M^{-1}]_{ii} = 0$ *for* $i = 1 \cdots r\}$.

We remark that $\mathrm{Galz}^0_\phi(R, G)$ is a subgroup of $\mathrm{Galz}_\phi(R, G)$ whose product $S \cdot T = (S \otimes_R T)^D$ is the usual product of Galois extensions. This follows by [CGO, (3.6)] and the fact that since $P = I$ the left and right G-actions on S in $\mathrm{Galz}^0_\phi(R, G)$ coincide.

PROOF. Since $A\phi A' - A$ is skew, $AMA' = A + A'$. Thus $PA' = A' - AMA' = -A$, $P^{-1}A = -A'$, so

$$P'MP = (I - MA')MP \quad \text{(since } M \text{ is symmetric)}$$

$$= MP - MA'MP = MP + MP^{-1}AMP = MP + MP^{-1}(I - P)P$$

$$= MP + MP^{-1}P - MP^{-1}PP = M.$$

Thus β is well defined. If $P = I$, $AM = 0$ and $A + A' = AMA' = 0$. Thus ν is well defined.

We show that the map β is an antihomomorphism.

Let S, T be in $\mathrm{Galz}(R, G)$ with left grading matrices P, Q, respectively. Then $ST = \Sigma\, S_\psi \otimes_\psi T$, where ψ runs through G^*. Since the right grading on ST is induced from that on T, and the left grading is induced from that on S, we can compute the matrix of ST by noticing that $T_{\chi_i} = {}_{\chi^{q_i}}T$, and $S_{\chi_i} = {}_{\chi^{p_i}}S$, so that $(ST)_{\chi_i} = S_{\chi^{q_i}} \otimes T_{\chi_i}$, and the left grade of

$$S_{\chi^{q_i}} = S_{\chi_1}^{q_{i1}} S_{\chi_2}^{q_{i2}} \cdots S_{\chi_r}^{q_{ir}} \text{ is } \chi^{p_1 q_{i1} + p_2 q_{i2} + \cdots + p_r q_{ir}} = \chi^{q_i P}.$$

So the left grading matrix of ST is QP.

If S, T have left grading matrices $P = Q = I$, then the G-grading on ST is that of $(S \cdot T)_{\chi_i} = S_{\chi_i} \otimes {}_{\chi_i}T = S_{\chi_i} \otimes T_{\chi_i}$, namely, $\sigma^{\alpha_i + \beta_i}$. So ν is a homomorphism.

If $\nu(S) = 0$ then S is trivially graded and $D = 0$, so S is commutative. On the other hand, it is easy to check that any trivially graded commutative Galois extension is in $\mathrm{Galz}^0(R, G)$. Thus the two sequences are exact. Since $P = I - AM$, if $M = 0, \beta = 1$. All that is left to prove are the ontoness statements for ν and β.

We begin showing ontoness of β, assuming M is invertible, by picking P in $O(M)$ and defining A by $A = (I - P)M^{-1}$. Then $P'MP = M$, $PM^{-1}P' = M^{-1}$, that is, $M^{-1} = (I - AM)M^{-1}(I - AM') = M^{-1} - A - A' + AMA'$, so $A\phi A' - A = D$ is skew. If $p = 2$ we assume D has zero diagonal. Thus associated to P are matrices A, D. We begin showing ontoness of ν by letting A be such that $AM = 0$ and $A' = -A$. Then associated to A are the skew matrix $D = A\phi A' - A$ and the nonsingular matrix $P = I = I - AM$. If $p = 2$ we assume D has zero diagonal.

Given P, A, D we define an algebra S in $\mathrm{Galz}_\phi(R, G)$ whose structure is described by these matrices as follows.

Let $S = \Sigma_\nu R u_1^{\nu_1} u_2^{\nu_2} \cdots u_r^{\nu_r}$, where $u_i^m = 1$. Fixing as above a primitive

mth root of unity ζ, define $u_i u_j = \zeta^{d_{ij}} u_j u_i$, extending by linearity. This defines the multiplication on S. Since $d_{ii} = 0$ it is well defined. Define gradings on S by giving u_i the left G^*-grading χ^{p_i}, the right G^*-grading χ^{ϵ_i}, and G-grading σ^{α_i}. It is then clear that S is a G-graded R-algebra and a Galois extension on the right with group G. Since P is invertible, one can find elements v_i in S so that v_i has left grading χ_i and the v_i generate S in the same way as the u_i. So S is a Galois extension on the left with group G. The other relations (1)–(4) of (1.1) all follow from the relationships among the matrices P, A, ϕ, D. This shows the ontoness of ν, and if M is invertible, the ontoness of β, completing the proof of the theorem.

(3.5) Remark. When R, G satisfy (3.1) the description of $\mathrm{Im}(\pi)$ for G a cyclic p-group given in [CGO, Theorem 4.1] follows easily from the techniques of Theorem (3.4) and the result, to be proved in §4, that, for G a cyclic p-group, $\mathrm{Im}(\pi) = \mathrm{Galz}(R, G)$. For suppose G is a cyclic p-group with generator σ, with the order of $G = p^e = m$. Choose ζ, a primitive p^eth root of unity, so that $\phi(\sigma, \sigma) = \zeta^{p^i}$ for some $i \geqslant 0$. Note that in applying (3.4) all matrices are 1×1.

For p odd, (3.4) reads:

$$1 \longrightarrow \mathrm{Comm}(R, G) \longrightarrow \mathrm{Galz}_\phi(R, G) \xrightarrow{\beta} O^1_{Z/p^eZ}((2p^i)) \longrightarrow 0$$

since $\mathrm{Skew}^1_{Z/p^eZ}((2p^i)) = 0$. Now $O^1_{Z/mZ}((2p^i)) = Z/2Z$ unless $\phi = 1$. If $i = 0, \beta$ is onto by (3.4) and we recover [CGO, 4.1]. If $i > 0$ then since $D = A\phi A' - A$ is skew, $D = 0$, so $A(\phi A' - I) = 0$, so $A = 0, P = I$ since $I - \phi A'$ is a unit. So if $i > 0, \beta = 1$ and we recover [CGO, 4.1].

For $p = 2, G$ of order $p^e, e > 1$, set $\phi(\sigma, \sigma) = \zeta^{2^i}$. Now $\{A \in \mathrm{Skew}^1(M) | A\phi A' - A = 0\} = \{0\}$, as is easily checked, so we simply have to compute β. We know $0 = D = A(\phi A - I)$ since cyclic Galois extensions are commutative. If $i \geqslant 1, \phi A - I$ is a unit and $A = 0, \beta = 1$. If $i = 0, A(A - I) = 0$ so $A = I$ or $A = 0$. If $A = I$, since $M = (2), P = -I$, otherwise $P = I$. So if $i \geqslant 1, \mathrm{Galz}_\phi(R, G) = \mathrm{Comm}(R, G)$; if $i = 0$ we get that $\mathrm{Im}(\beta) = Z/2Z$ and

$$1 \longrightarrow \mathrm{Comm}(R, G) \longrightarrow \mathrm{Galz}(R, G) \longrightarrow Z/2Z \longrightarrow 1$$

is exact, recovering [CGO, 4.1] in this case.

For $p = 2, G = Z/2Z, \phi(\sigma, \sigma) = (-1)^{2^i}$ ($i = 0$ or 1), and $M = 0$, so $\beta = 1$ and the sequence (3.4.2) describes $\mathrm{Galz}_\phi(R, G)$. Then it is easily checked that $\{A \in \mathrm{Skew}^1(M) | A\phi A' - A = 0\}$ is equal to $\{0\}$ if $i = 1$, i.e., ϕ is trivial, but equals $\{0, 1\}$ if $i = 0$, i.e., ϕ is nontrivial, thereby recovering [CGO, 4.1] in this case also.

4. **Ontoness of** π. In this section we prove, under appropriate conditions

on R, that π is onto. This result, together with those of §3, yields a description of $B_\phi(R, G)$ for any $G = \Pi Z_{p^e}$.

Throughout this section, the assumptions (3.1) hold, i.e., G is a group of order n and exponent m, and R is a connected commutative ring containing $1/n$ and a primitive mth root of unity.

We begin with some results on the structure of elements of $\mathrm{Galz}_\phi(R, G)$.

(4.1) LEMMA. *With R, G as above, let S be in $\mathrm{Galz}_\phi(R, G)$, and let $K = \{\sigma \in G | S_\sigma \neq 0\}$ be the group (2.8) of gradings of S: $S = \Sigma_{\sigma\in K} S_\sigma$. Let Z be the ungraded center of S and let $H = \{\sigma \in K | Z_\sigma \neq 0\}$. Then H is a group and $Z_\sigma = S_\sigma$ for all σ in H.*

PROOF. It suffices to verify both assertions when R is local. In that case, S, being a Galois extension of R with group G, is a twisted group ring $S = \Sigma_{\chi\in G^*} Ru_\chi$ with factor set in $U(R)$. The noncommutativity of S is expressed by $u_\chi u_\psi = D(\chi, \psi) u_\psi u_\chi$, where D: $G^* \times G^* \longrightarrow U(R)$ is skew and bilinear [6].

Now for any χ, ψ in $G^*, u_\chi u_\psi^{u_\chi} = \phi(u_\chi, u_\psi) D(\psi, \chi) u_\chi u_\psi$; if u_χ has grading σ_χ this becomes $\langle \psi, \sigma_\chi \rangle = \phi(\sigma_\chi, \sigma_\psi) D(\psi, \chi)$, that is, as a function of χ, D only depends on the grading of u. Thus, $H = \{\sigma \in K | \langle \psi, \sigma \rangle = \phi(\sigma, \sigma_\psi)$ for all ψ in $G^*\}$, a group, and if $\sigma \in H$, every u_χ in S_σ, hence all of S_σ (in view of (3.2)) is in Z.

(4.2) LEMMA. *If $S \in \mathrm{Galz}_\phi(R, G)$, R, G as above, if Z is the ungraded center of S and*

$$C = \{x \in S | x^\sigma = \phi(\sigma, x)x \text{ for all } \sigma \text{ in } G\},$$

then $Z = C \cdot S_1 \cong C \otimes_R S_1$.

PROOF. We know that S is a Galois extension of R with group G acting on the right, the image of σ in G on $s \in S$ being written s^σ. Define a new action of G on S by $\sigma(s) = \phi(\sigma^{-1}, s)s^\sigma$. Denote G when acting in that way on S by G_t. Notice that $C = \{s \in S | \sigma(s) = s \ \forall\ \sigma \in G_t\}$ by definition. Also, $\{\sigma | \sigma(s) = s$ for all $s \in S\} = H$. For if $\sigma \in G$ and $\sigma(s) = s$ for all s in S, then in particular $\sigma(s) = s$ for all s in S_1, but on $S_1, \sigma(s) = s^\sigma$, so $\sigma \in K$ by (2.8). Now for σ in K, x in $S_\sigma, xs^\sigma = \phi(\sigma, s)sx$, so $x\sigma(s) = sx$. Thus $\sigma(s) = s$ for all s in S iff for all x in S_σ, s in S, $xs = sx$, iff $\sigma \in H$, since H is the group of gradings of the center of S.

Now we observe that, for each σ in $(G/H)_t$, $\sigma(s) = s^\tau$ for some τ in G. For $\sigma(s) = \phi(\sigma^{-1}, s)s^\sigma$. We know from §3 that $S = \Sigma_{\chi\in G^*} S_\chi$ such that if $u_\chi \in S_\chi, \sigma \in G$, then $u_\chi^\sigma = \langle \chi, \sigma \rangle u_\chi$. Now $\sigma(u_\chi) = \phi(\sigma^{-1}, \mathrm{gr}(u_\chi)) \langle \chi, \sigma \rangle u_\chi$ (where $\mathrm{gr}(u_\chi)$ = the grade of u_χ). The map σ: $G^* \longrightarrow U(R)$ given by $\sigma(\chi) = \phi(\sigma^{-1}, \mathrm{gr}(u_\chi)) \langle \chi, \sigma \rangle$ is a homomorphism. Thus $\sigma \in \mathrm{Hom}(G^*, U(R)) =$

G, that is, there exists $\tau_\sigma \in G$ such that $\langle \chi, \tau_\sigma \rangle = \phi(\sigma^{-1}, \mathrm{gr}(u_\chi))\langle \chi, \sigma \rangle$ for all $\chi \in G^*$. This means that $\sigma(s) = s^{\tau_\sigma}$ for some τ_σ in G. Clearly this map $\sigma \mapsto \tau_\sigma$ is a homomorphism. Let L be the image in G of the map $\sigma \mapsto \tau_\sigma$. Then $L = (G/H)_t$, so that, with respect to the usual right action of G on S, C is the fixed ring of some subgroup L. Thus, by [4, (2.2)], S is a Galois extension of C with group $L = (G/H)_t$, and C is a Galois extension of R with group G/L, and ([4, (4.1)]) $\mathrm{rank}_R C = [G: L] = [H: 1]$.

Now in particular, C is $K/H^\perp \cap K = \mathrm{Gal}(Z/S_1)$-strong [4, (2.1)] since C is G-strong [4, (2.2)] and $H^\perp \cap K$ leaves C fixed. (Here K is as in (4.1) or (2.8) and $H^\perp = \{\sigma \in G \mid \phi(\sigma, \tau) = 1$ for all τ in $H\}$.) So the restriction of $K/H^\perp \cap K$ to C acts as a subgroup of $\mathrm{Gal}(C/R)$ with fixed ring $C \cap S_1$. But $C \cap S_1$ is easily seen to be R. So $\mathrm{Gal}(C/R)$ is the restriction of $K/H^\perp \cap K$ to C: $(K/H^\perp \cap K)/\bar{J} \cong \mathrm{Gal}(C/R)$, the isomorphism being by restriction, where $\bar{J} \subseteq K/H^\perp \cap K$ leaves C fixed. In fact, since $\mathrm{rank}_R C = \mathrm{rank}_{S_1} Z = [H: 1]$, $\bar{J} = \{1\}$. Now $(K/H^\perp \cap K)$ acts as a group of automorphisms of CS_1 with fixed ring S_1, and the Galois elements ([4, (1.3b)], or see §1) for C/R are at the same time Galois elements for S_1C/S_1. So CS_1 is a Galois extension of S_1 with group $(K/H^\perp \cap K)$. Also $C \otimes_R S_1$ is a Galois extension of S_1 with group $K/H^\perp \cap K$. Thus two applications of [4, (3.4)] to the maps $C \otimes_R S_1 \longrightarrow CS_1 \subseteq Z$ show that $C \otimes_R S_1 \cong Z$, proving the lemma.

We are ready for the main result of this section.

(4.3) THEOREM. *If R, G are as above, S is in $\mathrm{Galz}_\phi(R, G)$, H is the group of gradings of the center Z of S, and either H is a direct summand of G or* Pic(R) *is* $[H: 1]$*-torsion free, then $S \in \mathrm{Im}(\pi)$.*

PROOF. If $C = \{x \in S \mid x^\sigma = \phi(\sigma, x)x \ \forall\ \sigma \in G\}$, we know by (4.2) that $C \otimes_R S_1 \cong Z$, so for each σ in H, $C_\sigma \otimes_R S_1 \cong S_\sigma$. Thus C_σ is a rank one projective R-module, and from the equality $S_\sigma S_\tau = S_{\sigma\tau}$, one obtains that $(C_\sigma S_1)(C_\tau S_1) = (C_{\sigma\tau} S_1)$, i.e., $C_\sigma C_\tau \otimes_R S_1 = C_{\sigma\tau} \otimes_R S_1$; hence (since R is an an R-direct summand of S_1) $C_\sigma C_\tau = C_{\sigma\tau}$.

Thus if $H = \langle \tau_1 \rangle \times \cdots \times \langle \tau_r \rangle$, a product of cyclic groups, and H has order n_H, and if Pic(R) is n_H-torsion free, each C_τ is free and we may choose u_i, $i = 1 \cdots r$, so that $u_i u_j = u_j u_i$, $C_{\tau_i} = Ru_i$ and $u_i^{n_i} = a_i \in U(R)$ when $n_i =$ the order of τ_i. That is, $C = RH_f$, a commutative twisted group ring with twisting f, a symmetric 2-cocycle representing a class in $H^2(H, U(R))$. (A map f: $G \times G \longrightarrow U(R)$ is symmetric if $f(\sigma, \tau) = f(\tau, \sigma)$.)

We suppose, as we shall prove in (4.4), that there exists a symmetric 2-cocycle $g \in H^2(G, U(R))$ such that $\mathrm{res}_{G/H} g = f$, so that $C \cong RH_f \subseteq RG_g$ and RG_g is commutative. Let $A = RG_g \otimes_C S$ with multiplication $sz_\sigma = \phi(\sigma^{-1}, s)z_\sigma s^\sigma$; for

$z_\sigma \in C$, respectively (since RG_g is commutative) for $s \in C$, this agrees with the usual multiplication of C in S, respectively in RG_g. So the multiplication on A is well defined. We must show that A is separable over R, and for this it suffices to observe that C is a Galois extension of R, hence R-separable, and to prove that A is an Azumaya C-algebra. In fact, notice that if $\sigma(s) = \phi(\sigma^{-1}, s)s^\sigma$ is the G_t action on S defined in the proof of (4.2), then $sz_\sigma = z_\sigma\sigma(s)$. This suggests that A may be viewed as a crossed product $\Delta(S, (G/H)_t)_h$ with some factor set h: $G/H \times G/H \longrightarrow U(C)$; since S is a Galois extension of C with group $(G/H)_t$ (proof of (4.2)) it would follow that A is an Azumaya C-algebra split by S.

To show $A = \Delta(S, (G/H)_t)_h$ we observe that we can find an $h \in H^2(G/H, U(C))$ so that $RG_g \cong C(G/H)_h$. We do that by letting v: $G/H \longrightarrow G$ be a set splitting of the canonical map: $G \longrightarrow G/H$, with $v(\bar{1}) = 1$, and we define an R-module isomorphism ψ: $RG_g \longrightarrow C(G/H)$ induced by linearity and $\psi(\sigma) = (v(\bar{\sigma})^{-1}\sigma)\bar{\sigma} \in C(G/H)$. This is the identity on H, hence on C, and is an isomorphism of R-modules if it is 1-1. But $\psi_\sigma(\Sigma_{i\in G} r_i\sigma_i) = 0$ implies $\Sigma_{\sigma_i \in G} r_i(v(\bar{\sigma}_i)^{-1}\sigma_i)\bar{\sigma}_i = 0$, so for each $\bar{\sigma}$ in G/H, $\Sigma r_i(v(\bar{\sigma}_i)^{-1}\sigma_i) = 0$ where σ_i runs through the coset of $\sigma \bmod H$. If we set $\sigma_i = \tau_i\sigma, \tau_i \in H$, then this sum becomes $\Sigma_{h\in H}\, r_i(v(\bar{\sigma})^{-1}\sigma\tau_i) = 0$, i.e., $v(\bar{\sigma})^{-1}\sigma\Sigma r_i\tau_i = 0$. But $v(\bar{\sigma})^{-1}\sigma\tau_i$ are distinct elements of H for $\tau_i \in H$, so are linearly independent over R; hence, $r_i = 0$. Thus ψ is an isomorphism of R-modules. Define a product on $C(G/H)$, call it $C(G/H)_h$, to coincide with that on RG_g. Then ψ yields an isomorphism of C-algebras and $C(G/H)_h$ is a commutative twisted group ring with twisting h: $G/H \times G/H \longrightarrow C$. Then A is a crossed product as described. (In fact, A may be viewed as a smash product $S \#_C C[G/H]_h$ with respect to the $(G/H)_t$-action on S and the G/H-grading on $C[G/H]_h$.)

Grade $A = \Delta(S, (G/H)_t)_h = RG_g \otimes_C S$ diagonally. To show that $S = \pi(A)$ we must show $A^{A_1} = S, A^A = {}^AA = R$. First, $S \subseteq A^{A_1}$: for if $s_\sigma z_{\sigma^{-1}} \in A_1$,

$$ss_\sigma z_{\sigma^{-1}} = \phi(\sigma^{-1}, s)s_\sigma s^\sigma z_{\sigma^{-1}} = \phi(\sigma^{-1}, s)s_\sigma\phi(\sigma, s)z_{\sigma^{-1}}(s^\sigma)^{\sigma^{-1}} = s_\sigma z_{\sigma^{-1}}s.$$

If $z_\rho \in A^{A_1}$ then

$$\begin{aligned} z_\rho s_\sigma z_{\sigma^{-1}} &= s_\sigma z_{\sigma^{-1}} z_\rho = s_\sigma z_\rho z_{\sigma^{-1}} \quad \text{since } RG_g \text{ is commutative} \\ &= \phi(\rho^{-1}, s_\sigma) z_\rho s_\sigma^\rho z_{\sigma^{-1}} \end{aligned}$$

so $s_\sigma^\rho = \phi(\rho, s_\sigma)s_\sigma$. Since s_σ may be chosen at random from $hS, \rho \in H$, and so $z_\rho \in C \subseteq S$. Thus $A^{A_1} = S$. Now $A^A \subseteq A^{A_1} = S$. So

$$A^A \subseteq \{s \in S \mid z_\sigma s = \phi(\sigma, s) s z_\sigma \text{ for all } \sigma \text{ in } G\}$$

$$= \{s \in S \mid z_\sigma s = \phi(\sigma, s)\phi(\sigma^{-1}, s) z_\sigma s^\sigma\}$$

$$= \{s \in S \mid s = s^\sigma\} = R;$$

so $R = A^A$. Also ${}^A A \subseteq A^{A_1} = S$, and a similar argument shows ${}^A A \subseteq R$. Thus ${}^A A = R$.

We have therefore shown that whenever $C = RH_f \subseteq RG_g$, then such an A exists.

In case $G = H \times J$, $J = G/H$ and we can replace $RG_g = C((G/H)_t)_h$ by $C(J)_1$, the usual group ring of J. In that case it is not necessary to assume any conditions on Pic(R): $A = \Delta(S, J)_1$.

The proof of the theorem is complete as soon as we prove

(4.4) LEMMA. *Let G be a finite abelian group which acts trivially on the abelian group A. If H is a subgroup of G, then* res: $H^2_{\mathrm{sym}}(G, A) \longrightarrow H^2_{\mathrm{sym}}(H, A)$ *is onto.*

PROOF. We shall write the operation on A as addition. We denote by $H^2_{\mathrm{sym}}(G, A)$ the subgroup of $H^2(G, A)$ consisting of classes represented by symmetric cocycles, those cocycles f: $G \times G \longrightarrow A$ satisfying $f(\sigma, \tau) = f(\tau, \sigma)$. Note that coboundaries are always symmetric. From [20, Theorem 2.1 and p. 159] we know that if $H = W_1 \times \cdots \times W_r$, then

$$H^2_{\mathrm{sym}}(H, A) \cong \bigoplus_{i=1}^{r} H^2_{\mathrm{sym}}(W_i, A)$$

the isomorphism being induced by restriction. Similarly for G. Thus it suffices to prove the lemma assuming H is cyclic of order $p^f = c$ (p prime) with generator τ, and we can assume G is a p-group.

Let $G = Z_1 \times \cdots \times Z_n$ where $Z_i = \langle \sigma_i \rangle$ is cyclic of order $d_i = p^{e_i}$. Let $\tau = \sigma_1^{r_1} \cdots \sigma_n^{r_n}$. If τ has order p^f then $\sigma_i^{r_i}$ has order exactly $p^f = c$ for some i. Fix such an i. Replacing $\sigma_i^{r_i}$ by $\sigma_i^{ur_i}$ where u is some appropriate integer relatively prime to p, we can assume that $r_i c = d_i$. Now $A/cA \cong H^2(H, A)$: an isomorphism is given by sending the class of a to the class of $f_{a,\tau}$, where, setting $[r]$ = the greatest integer $\leqslant r$,

$$f_{a,\tau}(\tau^\lambda, \tau^\mu) = a[(\lambda + \mu)/c].$$

(That is, $f_{a,\tau}$ is the cup product of $a + cA$ in $\hat{H}^0(H, A)$ (Tate cohomology) with the class of $\delta\chi_\tau$ in $H^2(H, \mathbf{Z})$ where χ_τ: $H \longrightarrow \mathbf{Q}/\mathbf{Z}$ is given by $\chi_\tau(\tau^\lambda) = \lambda/n$—see [16, p. 141].) So it suffices to show that $f_{a,\tau}$: $H \times H \longrightarrow A$ is the restriction of a 2-cocycle g on G. We choose such a g: $G \times G \longrightarrow A$ as follows:

$$g(\sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}, \sigma_1^{\beta_1} \cdots \sigma_n^{\beta_n}) = g(\sigma_i^{\alpha_i}, \sigma_i^{\beta_i}) = a[(\alpha_i + \beta_i)/d_i].$$

Then

$$\operatorname{res} g(\tau^\lambda, \tau^\mu) = g(\sigma_i^{r_i\lambda}, \sigma_i^{r_i\mu}) = a\left[\frac{r_i\lambda + r_i\mu}{d_i}\right] = a\left[\frac{\lambda + \mu}{c}\right] = f_{a,\tau}(\tau^\lambda, \tau^\mu).$$

So $\operatorname{res} g = f_{a,\tau}$. Also, g is a 2-cocycle. For g is a 2-cocycle iff $\operatorname{res}_{G \to Z_i} g$ is a cocycle in $H^2(Z_i, A)$, clearly; and $\operatorname{res}_{G \longrightarrow Z_i} g = f_{a,\sigma_i}$ can be shown to be a cocycle either directly or by identifying it as the image of a cup product, as above. That completes the proof of (4.4) and (4.3).

5. **Dimodule algebras.** In this section we note the relationship between $B_\phi(R, G)$ and a Brauer group $BD(R, G)$ of G-dimodule algebras defined by F. W. Long [12], [13] and apply the results of the previous sections to $BD(R, G)$.

Assume G is a finite abelian group, and $H = RG$. A G-dimodule algebra A is a G-graded algebra and a G-module algebra such that the G-action preserves the grading, that is, the following diagram [13, Definition 3.1(i)] commutes:

$$\begin{array}{ccc} H \otimes A & \xrightarrow{\quad \mu \quad} & A \\ \big\downarrow {\scriptstyle 1 \otimes \Delta} & & \big\downarrow {\scriptstyle \Delta} \\ H \otimes A \otimes H & \xrightarrow{\quad \mu \otimes 1 \quad} & A \otimes H \end{array}$$

Let H^* be the linear dual of H. We may define an H^*-module structure on A from the H-grading on A via the action $h^*(a) = (1 \otimes h^*)\Delta(a)$.

This H^*-module structure and the H-module structure commute, since they operate on different factors of $\Delta(a)$ for a in A. Thus A is an $H^* \otimes H$-module. If A is a G-graded algebra, then it is an H-comodule algebra, hence [13, Remark following Definition 2.9] an H^*-module algebra. Thus A is an $H^* \otimes H$-module algebra. Thus A is an $H \otimes H^*$-comodule algebra.

So an H-dimodule algebra is an $H \otimes H^*$-comodule algebra. The converse is equally clear.

Given two G-dimodule algebras A and B their product is the smash product $A \# B$, with multiplication given on homogeneous elements by

$$(a_1 \# b_1)(a_2 \# b_2) = a_1 {}^{b_1}a_2 \# b_1 b_2,$$

where if b_1 is homogeneous of grade σ, ${}^{b_1}a_2 = {}^{\sigma}a_2$. In terms of the $(RG)^*$-comodule structure of A,

$$^{\sigma}a = \sum_{(a)} \langle \sigma, a_{(2)} \rangle a_{(1)}.$$

Suppose the order of G is a unit in R, and $\mathrm{Hom}(G, U(R)) = G^* \cong G$, so that $(RG)^* = R(G^*)$. Then $A = \Sigma_{\chi \in G^*} A_\chi$ such that, for $a \in A_\chi$, $\Delta(a) = a \otimes u_\chi$, and the smash product on homogeneous elements becomes

$$(a_1 \# b_1)(a_2 \# b_2) = a_1 a_2 \langle G\text{-grade}(b_1), G^*\text{-grade}(a_2) \rangle \# b_1 b_2.$$

Let ϕ: $(G \times G^*) \times (G \times G^*) \longrightarrow U(R)$ be the bilinear map defined by

$$\phi((\sigma_1, \chi_1), (\sigma_2, \chi_2)) = \langle \sigma_1, \chi_2 \rangle.$$

Then $A \# B = A \otimes B$, an instance of the graded tensor product of [CGO].

It is easy to check that G-Azumaya in the sense of Long and graded Azumaya in the sense of [CGO] coincide by observing that the maps F, G of [12, Definition following Theorem 1.3] coincide with the maps η_A, μ_A of [CGO, (2.8)].

Since the Azumaya algebras which are trivial in [CGO, (2.10)] and in [12, Definition preceding Theorem 1.5] coincide, we have, summing up:

(5.1) THEOREM. *If R is a connected commutative ring, G is an abelian group of order n and exponent m, and R contains $1/n$ and a primitive mth root of unity, then Long's Brauer group $BD(R, G)$ is equal to $B_\phi(R, G \times G^*)$ $\cong B_\phi(R, G \times G)$ for ϕ as above.*

We note that when G is a p-group whose r cyclic direct summands all have equal order, then $G \times G$ is also, and the matrix M arising in §3 corresponding to the ϕ of the theorem is nonsingular—it is $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ where I is the $r \times r$ identity matrix.

Summing up our results as applied to Long's group $BD(R, G)$ we have

(5.2) THEOREM. *If $G = \Pi_p(\Pi_{i=1}^{r_p} Z_{p^{e_p}})$ has order n, and R is a connected commutative ring containing $1/n$ and a primitive p^eth root of unity for each $p|n$ and* Pic(R) *is p-torsion free for all $p|n$, then $BD(R, G)$ is described by the two exact sequences*

$$0 \longrightarrow B(R) \longrightarrow BD(R, G) \xrightarrow{\pi} \mathrm{Galz}_\phi(R, G \times G) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Comm}(R, G \times G) \longrightarrow \mathrm{Galz}_\phi(R, G \times G) \xrightarrow{\beta} \prod_p O^{2r_p}_{Z/p^eZ}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where β is onto $O^{2r_p}_{Z/p^eZ}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ for p odd, and the image of β in $O^{2r_2}_{Z/2^eZ}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ is

$$\left\{P \,\middle|\, \left[(I-P)\begin{pmatrix}0 & 0\\ I & 0\end{pmatrix}(I-P') - (I-P)\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\right]_{ii} = 0 \text{ for all } i = 1, \cdots, 2r_2\right\}.$$

The assumption that Pic(R) *is* p*-torsion free may be omitted if* $r_p = 1$ *and* p *is odd, or if* $e_p = 1$.

PROOF. All but the last statement follows from what we have already done.

If $e_p = 1$ then $G_p = \Pi_{i=1}^{r_p} Z_p$ and every subgroup is a direct summand, so by Theorem (4.3) the assumption on Pic(R) is not needed.

If $r_p = 1$ and p is odd the p part of the theorem states that for $G = Z_{p^e}$

$$0 \longrightarrow \mathrm{Comm}(R, G \times G) \longrightarrow \mathrm{Galz}_\phi(R, G \times G) \longrightarrow O^2_{Z_{p^e}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \longrightarrow 0$$

is exact.

We show π is onto by showing that for each S in Galz(R, $G \times G$), the gradings H of the center are a direct summand of $G \times G$. We do this by explicitly computing $O^2_{Z/p^eZ}\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$.

Let $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in O^2_{Z/p^eZ}\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$. Then

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a & c\\ b & d\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

so $2ab = 0 = 2cd$, $ad + bc = 1$. From $ab = 0$, we get, in Z, $a = up^r$, $b = vp^s$ with $r + s \geqslant e$, $(uv, p) = 1$. But from $ad + bc = 1$, either $r = 0$ or $s = 0$. Similarly for c and d. So we have only two cases.

Case 1. $a = u$, $d = u^{-1}$, $c = b = 0$ in Z/p^eZ.

Case 2. $b = u$, $c = u^{-1}$, $a = d = 0$. We use the notation of §3.

Case 1.

$$P = \begin{pmatrix}u & 0\\ 0 & 1/u\end{pmatrix}, \quad A = \begin{pmatrix}0 & 1-u\\ 1-1/u & 0\end{pmatrix}, \quad D = \begin{pmatrix}0 & u-1\\ 1-u & 0\end{pmatrix} = A\begin{pmatrix}-u & 0\\ 0 & -1\end{pmatrix}.$$

The center is

$$\{rA \mid r \in Z^2_{p^e}, rD = 0\} = \left\{rA \,\middle|\, rA\begin{pmatrix}-u & 0\\ 0 & -1\end{pmatrix} = 0\right\} = \{0\}$$

since $\left(\begin{smallmatrix}u & 0\\ 0 & 1\end{smallmatrix}\right)$ is invertible.

Case 2.

$$P = \begin{pmatrix}0 & u\\ 1/u & 0\end{pmatrix}, \quad A = \begin{pmatrix}-u & 1\\ 1 & -1/u\end{pmatrix}, \quad D = 0.$$

So the center is $\{rA \mid r \in Z^2_{p^e}\}$ = the row space of A. But the row space of A is isomorphic to Z_{p^e}, so is a direct summand of $G \times G$. That proves the case $r_p = 1, p$ odd.

(5.3) REMARK. Observe that in this last proof we obtained essentially that $O^2_{Z/p^eZ}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong D_{p^e - p^{e-1}}$, the dihedral group, where $p^e - p^{e-1}$ is the number of units in Z/p^eZ (cf. [12, Theorem 2.7]). From (5.2) it follows that $BD(\mathrm{C}, Z/p^eZ) \cong D_{p^e - p^{e-1}}$. The complexity of $O^{2r_p}_{Z/p^eZ}$, and hence of $BD(\mathrm{C}, (Z/p^eZ)^r)$ is much greater if $r_p > 1$. For example, for $p = 3, n = 2$, $e = 2$, there is a short exact sequence

$$0 \longrightarrow (Z/3Z)^6 \longrightarrow O^4_{Z/9Z}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \longrightarrow O^4_{Z/3Z}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \longrightarrow 0.$$

$O^4_{Z/3Z}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ has a subquotient isomorphic to $PSL_2(Z/3Z) \times PSL_2(Z/3Z)$ [1] and the whole group has order $3^8 \times 2^7$. When the target of π is so complicated a direct approach to showing that' π is onto (as used in the proof of the last statement of (5.2)) has proved unfruitful.

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